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WAVELESS GRAVITY FLOW OVER AN INCLINED STEP

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Waveless gravity flows over an inclined step of height H are studied. The cases $H > 0$ and $H < 0$ are considered. It is always assumed that the flow is supercritical at infinity on the right, which ensures the existence of a solution with no downstream wave. For the case of subcritical flow, the relation between the Froude number and the step height is determined that ensures a waveless regime. An approximate analytical relation between the step height and the Froude number is obtained from an analysis of numerical data. This dependence is shown to be nearly identical for steps of any slopes. For the case of supercritical flow, it is established that the problem has a two-parameter set of solutions. For the case $H < 0$, approximate analytical formulas for the free-surface shape are obtained.

Key words: *gravity fluid, flow over a step, subcritical and supercritical regimes.*

Introduction. The problem of gravity flow above an uneven bottom is generally formulated as follows: an ideal incompressible fluid moves above an above uneven bottom of specified shape. The motion is considered steady-state and irrotational. At infinity on the left, the bottom is asymptotically horizontal and the layer width h and unperturbed-flow velocity V_0 are specified. The free-surface shape above the bottom is unknown beforehand and is to be determined during the solution. At infinity on the right, the boundary conditions depend on the features of the problem. For example, if the bottom is asymptotically horizontal downstream, it is assumed that the velocity is bounded at infinity on the right. The main parameter describing such flows is the Froude number $\mathrm{Fr} = V_0/\sqrt{gh}$, where g is the acceleration of gravity. The velocity $V_* = \sqrt{gh}$ is called critical. For $\mathrm{Fr} < 1$, the flow is subcritical, and for $\mathrm{Fr} > 1$, it is supercritical.

The branch of hydromechanics dealing with free-surface flows over obstacles has been actively developed. A bibliography of studies in this direction can be found in [1–6].

For any flows over a step with free surfaces asymptotically horizontal at infinity on the left and on the right, the following formulas are valid [7]:

$$\mathrm{Fr}^2 + 2 = \mathrm{Fr}^2 /(L - H/h)^2 + 2L; \tag{1}$$

$$\mathrm{Fr}(\infty) = \mathrm{Fr} /(L - H/h)^{3/2}. \tag{2}$$

Here $L = y(\infty)/h$, $y(\infty)$ is the ordinate of the free surface at infinity on the right; $\mathrm{Fr}(\infty)$ is the Froude number at infinity on the right; and H is the step height.

For $H > 0$, the problem of flow over a step can be treated as the problem of spillway with a wide crest. If Q is the spillway flow rate, $\mathrm{Fr} = Q/\sqrt{gh^3}$. Thus, for fixed g and h, the Froude number is directly proportional to the flow rate. In hydraulics, the spillway flow rate is determined using the so-called maximum flow rate principle (MFRP), according to which a waveless spillway flow regime with a maximum flow rate is established by itself with time (see [8]). We use the MFRP to approximately determine the relationship between Fr and H/h. We assume that Eq. (1) implicitly specifies the dependence $\mathrm{Fr} = \mathrm{Fr}(L)$ for fixed values of H/h. Differentiating (1) with respect to L, we obtain

$$2\,\mathrm{Fr}\,\mathrm{Fr}'_L = \frac{2\,\mathrm{Fr}\,\mathrm{Fr}'_L(L - H/h) - 2(L - H/h)\,\mathrm{Fr}^2}{(L - H/h)^4} + 2.$$

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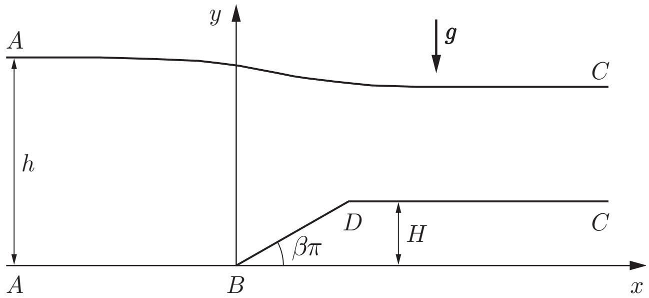

Fig. 1. Physical flow region.

Substituting $\mathrm{Fr}'_L = 0$ into this formula, we obtain

$$\mathrm{Fr}^2/(L - H/h)^3 = \mathrm{Fr}^2(\infty) = 1. \tag{3}$$

This relation implies that

$$L = H/h + \mathrm{Fr}^{2/3}, \tag{4}$$

and the desired relationship between Fr and H/h follows from (1):

$$H/h = 1 + \mathrm{Fr}^2/2 - 3\,\mathrm{Fr}^{2/3}/2. \tag{5}$$

Relation (5) is approximate, and one of the objectives of the present study is to refine the relation between Fr and H/h.

Formulation of the Problem and Its Reduction to a Nonlinear Boundary-Value Problem for an Analytic Function. We consider the steady-state potential flow of an ideal incompressible gravity fluid above an uneven polygonal bottom in the shape of an inclined step (Fig. 1). Cartesian coordinates xy are introduced with origin in the base of the step. The following parameters are specified: the unperturbed free-surface depth at infinity on the left h, the incident-flow velocity V_0, the slope of the step to the x axis $\beta\pi$, and the acceleration of gravity g. Gravity is oppositely directed to the y axis.

We map the flow region in the physical plane $z = x + iy$ onto the strip $D_t = \{0 < \eta < \pi/2\}$ in the plane of the complex variable $t = \xi + i\eta$, so that infinitely remote points of the physical plane are transformed to infinitely remote points of the strip, and the corner points of the bottom B and D to the points $t = 0$ and $t = d$, respectively.

We introduce the analytic function

$$\chi(t) = \ln\Big(\frac{\pi}{2h}\,\frac{dz}{dt}\Big) = r + i\Theta,$$

where Θ is the angle between the velocity vector and the x axis. Once the function $\chi(t)$ is found, the desired conformal mapping $z(t)$ is given by the integral

$$z(t) = \frac{2h}{\pi}\int_0^t \exp\,(\chi(t))\,dt. \tag{6}$$

The flow region in the plane of the complex potential $W = \varphi + i\psi$ corresponds the strip $D_t = \{0 < \psi < V_0 h\}$. The relationship between the region of the complex potential and the region t is defined by the expression

$$W = 2hV_0 t/\pi.$$

We introduce the notation

$$\operatorname{Re}\chi(t) = r^0(\xi), \quad \operatorname{Im}\chi(t) = \Theta^0(\xi), \qquad t = \xi, \quad -\infty < \xi < \infty,$$
$$\operatorname{Re}\chi(t) = r^1(\xi), \quad \operatorname{Im}\chi(t) = \Theta^1(\xi), \qquad t = \xi + i\pi/2, \quad -\infty < \xi < \infty$$

and formulate the boundary-value problem for the function $\chi(t)$ analytic in the region D_t. From (6) it follows that on the upper boundary of the strip D_t, the following equality is valid:

$$\frac{dz}{dt} = \frac{2h}{\pi}\,\exp\,(r^1)(\cos\Theta^1 + i\sin\Theta^1).$$

Since $y'_\xi = \operatorname{Im} dz/dt$, we have

$$y'_\xi = (2h/\pi)\exp(r^1)\sin\Theta^1. \tag{7}$$

For the complex conjugate velocity dW/dz, the following formula is valid:

$$\frac{dW}{dz} = \frac{dW}{dt}\,\frac{dt}{dz} = V_0 \exp(-\chi(t)). \tag{8}$$

Let V be the velocity on the free surface. Then, Eq. (8) implies that

$$V = V_0 \exp(-r^1). \tag{9}$$

On the free surface, Bernoulli's relation should hold:

$$V^2 + 2gy = \text{const}.$$

Differentiating this relation with respect to ξ, we obtain

$$VV'_\xi + gy'_\xi = 0.$$

Substituting the previously obtained expressions for y'_ξ and V into the last formula, we find that for $\eta = \pi/2$ on the free surfaces, the function $\chi(t)$ satisfies the condition

$$\frac{dr^1}{d\xi} = \frac{2}{\pi\,\mathrm{Fr}^2}\exp(3r^1)\sin\Theta^1. \tag{10}$$

For $\eta = 0$ at the bottom, the function $\chi(t)$ satisfies the conditions

$$\Theta^0 = \begin{cases} 0, & \xi \leqslant 0,\ \xi \geqslant d, \\ \beta\pi, & 0 < \xi < d. \end{cases} \tag{11}$$

At infinity on the left, we have $V \to V_0$; therefore, by virtue of (8),

$$\chi(-\infty) = 0. \tag{12}$$

At infinity on the right, the velocity V is limited; therefore

$$|\chi(\infty)| < \infty. \tag{13}$$

Because expressions (11) contain the unknown parameter d that specifies the position of the point D in the parametric plane, it follows that to close the problem, we need to derive an additional condition. In the derivation of the this condition, the flow regime over the step is of significance.

Subcritical Flow. We shall study waveless flow regimes in which subcritical flow becomes supercritical. Because $\mathrm{Fr} > 1$ at infinity on the right, waves should be absent there. However, at infinity on the left, $\mathrm{Fr} < 1$, and there waves can arise. Hence, it is necessary to derive the condition of no waves at infinity on the left.

We assume that at infinity on the left, the function $\chi(t)$ behaves as follows:

$$\chi(t) \sim A\exp(kt) \qquad \text{at} \quad \xi \to -\infty. \tag{14}$$

Here $A = \text{const}$ and $k > 0$. Linearizing boundary condition (10), we obtain

$$\frac{dr^1}{d\xi} - \frac{2}{\pi\,\mathrm{Fr}^2}\Theta^1 = 0. \tag{15}$$

From relation (14), we find that

$$r^1 = A\exp(k\xi)\cos(\pi k/2), \qquad \Theta^1 = A\exp(k\xi)\sin(\pi k/2).$$

Substituting the obtained expressions for r^1 and Θ^1 into (15), we find the relation linking the exponent k and the Froude number:

$$\mathrm{Fr}^2\,\pi k/2 = \tan(\pi k/2). \tag{16}$$

An analysis of the functions $U = \tan(\pi k/2)$ and $U = \mathrm{Fr}^2\,\pi k/2$ shows that for $\mathrm{Fr} > 1$, the first positive root of Eq. (16) is on the segment $(0, 1)$, and for $\mathrm{Fr} \leqslant 1$, it is on the segment $(2, 3)$. Thus, the exponent $k > 1$ for subcritical flow and $k < 1$ for supercritical flow. Physically, this implies that the free surface at infinity on the left approaches the horizontal asymptote more rapidly for subcritical flow than for supercritical flow.

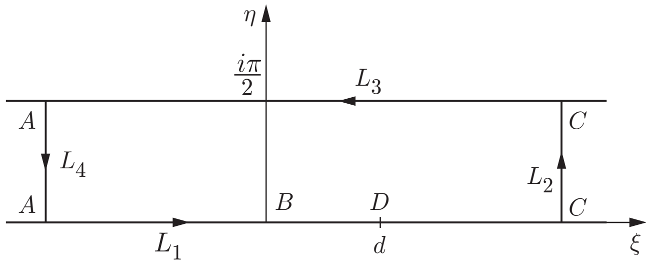

Fig. 2. Parametric region.

We consider the function $\Phi(t) = \chi(t)\exp(-t)$. Let $\mathrm{Fr} < 1$, then $k > 1$ and the function $\Phi(t)$ can be integrated over the boundary of the strip D_t. In this case, the infinitely remote points are obtained from the vertical segments of the straight lines L_2 and L_4 (Fig. 2).

According to Cauchy's theorem,

$$\mathrm{Im}\Big[\int\limits_{L_1}\Phi(t)\,dt+\int\limits_{L_2}\Phi(t)\,dt+\int\limits_{L_3}\Phi(t)\,dt+\int\limits_{L_4}\Phi(t)\,dt\Big]=0. \tag{17}$$

Since $|\chi(\infty)| < \infty$ and $\exp(-t) \to 0$ as $t \to +\infty$, the following equality holds:

$$\int\limits_{L_2}\chi(t)\exp(-t)\,dt=0.$$

At infinity on the left, $\chi(t) \simeq A\exp(kt)$, where $k > 1$; hence, $\chi(t)\exp(-t) \to 0$ as $t \to -\infty$. This implies that

$$\int\limits_{L_4}\chi(t)\exp(-t)\,dt=0.$$

Integration along the segment L_1 yields

$$\mathrm{Im}\Big[\int\limits_{-\infty}^{+\infty}\exp(-\xi)(r^0+i\Theta^0)\,d\xi\Big]=\int\limits_{-\infty}^{+\infty}\Theta^0\exp(-\xi)\,d\xi=\int\limits_{0}^{d}\beta\pi\exp(-\xi)\,d\xi=\beta\pi(1-\exp(-d)).$$

Integration along the segment L_3 yields

$$\mathrm{Im}\Big[\int\limits_{-\infty}^{+\infty}\exp(-(\xi+i\pi/2))(r^1+i\Theta^1)\,d\xi\Big]=-\int\limits_{-\infty}^{+\infty}r^1\exp(-\xi)\,d\xi.$$

To the integral obtained, we apply the partial integration formula:

$$-\int\limits_{-\infty}^{+\infty}r^1\exp(-\xi)\,d\xi=r^1\exp(-\xi)\Big|_{-\infty}^{+\infty}-\int\limits_{-\infty}^{+\infty}\exp(-\xi)\,\frac{dr^1}{d\xi}\,d\xi=-\int\limits_{-\infty}^{+\infty}\exp(-\xi)\,\frac{dr}{d\xi}\,d\xi.$$

Substitution of the obtained integrals into (17) yields the condition that closes the problem:

$$\int\limits_{-\infty}^{+\infty}\frac{dr^1}{d\xi}\,\exp(-\xi)\,d\xi=\beta\pi(\exp(-d)-1). \tag{18}$$

Conditions (10)–(13) and (18) define the boundary-value problem of finding a function $\chi(t)$ analytic in the region D_t and the parameter d, where d is the step height in the plane t. Thus, in the case of subcritical flow, the step height is not specified but is determined from the condition of no waves at infinity on the left.

Supercritical Flow. We now consider a supercritical flow over the step. In this case, we assume that supercritical flow on the left becomes supercritical flow on the right. We designate $q = L - H/h$ and write Eq. (1) as

$$\mathrm{Fr}^2+2-\mathrm{Fr}^2/q^2-2q-2H/h=0.$$

Transforming it to a cubic equation for q, we have

$$q^3 - q^2(\mathrm{Fr}^2/2 + 1 - H/h) + \mathrm{Fr}^2/2 = 0. \tag{19}$$

From physical considerations, it is obvious that $q > 0$, and, hence, only the positive roots of Eq. (19) are of interest. Let

$$q_* = (2/3)(\mathrm{Fr}^2/2 + 1 - H/h).$$

From Bernoulli's equations, it follows the free-surface elevation above the horizontal bottom level cannot be higher than $(\mathrm{Fr}^2/2 + 1)h$. Therefore, $q_* > 0$. We introduce the function

$$P(q) = q^3 - q^2(\mathrm{Fr}^2/2 + 1 - H/h) + \mathrm{Fr}^2/2.$$

For $q \geqslant 0$, this function reaches the minimum value at the point $q = q_*$:

$$P_{\min} = \mathrm{Fr}^2/2 - (4/27)(\mathrm{Fr}^2/2 + 1 - H/h)^3.$$

The equation $P(q) = 0$ has two positive roots for $P_{\min} < 0$ or a multiple root $q = q_{\min}$ for $P_{\min} = 0$ or it does not have roots for $P_{\min} > 0$. Hence, for the resolvability of the problem in question, it is necessary that $P_{\min} \leqslant 0$. The last condition can be written as

$$\mathrm{Fr}^2/q_*^3 \leqslant 1 \tag{20}$$

or

$$H/h \leqslant \mathrm{Fr}^2/2 + 1 - 3\,\mathrm{Fr}^{2/3}/2. \tag{21}$$

Thus, the Froude number (Fr) and the step height (H/h) cannot be specified arbitrarily but they should be specified so that inequality (21) be satisfied. We note that the equality in (21) exactly corresponds to the Froude number determined using the MFRP (see [6]).

Let condition (20) be satisfied. We denote the positive roots of Eq. (19) by q_1 and q_2. Then, $q_1 < q_* < q_2$ and, hence,

$$\frac{\mathrm{Fr}^2}{q_2^3} < \frac{\mathrm{Fr}^2}{q_*^3} < \frac{\mathrm{Fr}^2}{q_1^3}.$$

From this and from inequality (20) it follows that $\mathrm{Fr}^2/q_2^3 < 1$. However, according to formula (2),

$$\mathrm{Fr}^2/q^3 = \mathrm{Fr}^2(\infty).$$

Thus, if $q = q_2$, then $\mathrm{Fr}(\infty) < 1$ in all cases. Hence, the largest (in magnitude) solution of Eq. (19) always corresponds to subcritical flow at infinity on the right. However, because we consider flows for $\mathrm{Fr}(\infty) > 1$, then in the solution of Eq. (19), it is necessary to choose the root of the smallest magnitude.

Solving Eq. (19), we find q and $y(\infty) = qh + H$ and taking into account that $V_\infty = V_0 \exp(-r^1(\infty))$, we write Bernoulli's relation at infinitely remote points of the free surface in the form

$$V_0^2 + 2gh = V_0^2 \exp(-2r^1(\infty)) + 2gy(\infty).$$

From this,

$$r^1(\infty) = -(1/2)\ln[1 - (q + H/h - 1)/\mathrm{Fr}^2]. \tag{22}$$

We now have conditions (10)–(13) and (22) to determine the function $\chi(t)$ and the parameter d.

Thus, in the case of transition of supercritical flow on the left to supercritical flow on the right, it is possible to specify the step height in the physical plane H.

We note the difference between the closing conditions (18) and (22). They both serve to determine the mathematical parameter d and were derived under the assumption that the flow regime is waveless. However, in the subcritical case for a fixed Froude number, one obtains the parameter d from Eq. (18) and then the value of the parameter H/h corresponding to this d. In the supercritical case, the obtained value of d ensures the beforehand specified value of H/h.

Reducing the Problem to Solving a System of Nonlinear Integral Equations. We introduce the function $\lambda(\xi) = dr^1/d\xi$ and assume that this function is known. From this it follows that

$$r^1(s) = \int\limits_{-\infty}^{s} \lambda(\xi)\,d\xi = C[\lambda](s), \tag{23}$$

where $C[\lambda]$ is a linear integral operator.

Relations (11) and (23) are the conditions of the mixed boundary-value problem for the function $\chi(t)$ analytic in the strip D_t: the real part of $\chi(t)$ is specified on the upper boundary of the strip, and the imaginary part on the lower part. The solution of this problem is given by the formula (see, e.g., [9])

$$\chi(t) = \frac{1}{\pi}\Big[\int\limits_{-\infty}^{+\infty} \frac{r^1(\xi)\,d\xi}{\cosh\,(\xi - t)} + \int\limits_{-\infty}^{+\infty} \frac{\Theta^0(\xi)\,d\xi}{\sinh\,(\xi - t)}\Big].$$

Taking into account boundary conditions (11), for $\Theta^0(\xi)$ we can write

$$\chi(t) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \frac{r^1(\xi)\,d\xi}{\cosh\,(\xi - t)} + \alpha\Big(\ln\frac{1-\exp\,(t-d)}{1+\exp\,(t-d)} - \ln\frac{1-\exp\,(t)}{1+\exp\,(t)}\Big). \tag{24}$$

From formula (24) it is easy to obtain the relationship between the real part $r^1(\xi)$ and imaginary part $\Theta^1(\xi)$ of the function $\chi(t)$ on the upper boundary of the strip D_t

$$\Theta^1(\xi) = B[r^1](\xi) + \lambda_2(\xi), \tag{25}$$

where

$$\lambda_2(\xi) = 2\beta(\arctan\,(\exp\,(d-\xi)) - \arctan\,(\exp\,(-\xi))),$$

and $B[r^1]$ is a linear singular operator defined as

$$B[r^1](s) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \frac{r^1(\xi)\,d\xi}{\sinh\,(\xi - s)}. \tag{26}$$

The integral in (26) is a principal-value integral.

We introduce a superposition of the operators B and C: $D[\lambda] = BC[\lambda]$ and write boundary condition (10) as the nonlinear integral equation

$$\lambda(\xi) = (2/(\pi\,\mathrm{Fr}^2))\exp\,(3C[\lambda])\sin\,(D[\lambda] + \lambda_2(\xi)). \tag{27}$$

Relations (18) and (22) are expressed in terms of the function $\lambda(\xi)$ as follows:

$$E[\lambda(\xi)\exp\,(-\xi)] = \beta\pi(\exp\,(-d) - 1); \tag{28}$$

$$E[\lambda] = -(1/2)\ln\,[1 - (q + H/h - 1)/\,\mathrm{Fr}^2]. \tag{29}$$

Here $E[\lambda]$ is a linear functional of the form

$$E[\lambda] = \int\limits_{-\infty}^{+\infty} \lambda(\xi)\,d\xi. \tag{30}$$

In relations (27)–(29), it is necessary to determine the function λ and the parameter d. System (27), (28) is solved in the case of subcritical flow, and system (27) and (29) is solved in the case of a supercritical flow.

Systems (27), (28) and (27), (29) are nonlinear and are solved numerically using the Newton method. A digitization method for systems of this type is described in [10, 11].

Calculation Results. *Subcritical Flow.* In the case where subcritical flow on the left becomes supercritical on the right, a waveless flow regime is possible only if there is a definite relation between the step height and the Froude number. In the case of using the hydraulic maximum flow rate principle, this relation is specified by formula (4) and the function $H/h(\mathrm{Fr})$, according to the MFRP, does not depend on the slope of the step.

Calculations for a vertical step with arbitrary values $0 < \mathrm{Fr} < 1$ and H/h were performed in [6]. According to these calculations, an infinite train of periodic waves occurs at infinity on the right. If the values of $H/h > 0$ are fixed and the Froude number is increased, the wavelength increases. For a certain value of Fr, the wavelength becomes infinite and the wave train behind the step degenerates into a train of "solitary waves." In this case, the crest of the first wave behind the step goes to infinity and the flow regime becomes waveless. Thus, according to the calculations of [6], the MFRP is strictly valid in the sense that waveless flow regimes corresponds to maximum flow rates (Froude number). However, it should be noted that in this treatment of the MFRP, the maximum flow

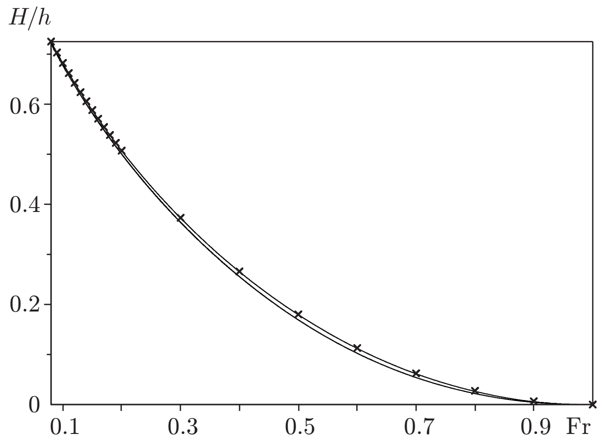

Fig. 3. Curves of H/h versus Fr for $\beta = 0.2$, 0.3, 0.4, and 0.5. The data obtained using the MFRP are denoted by points.

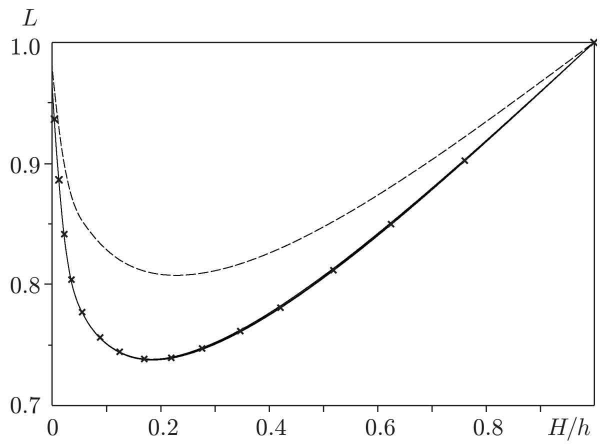

Fig. 4

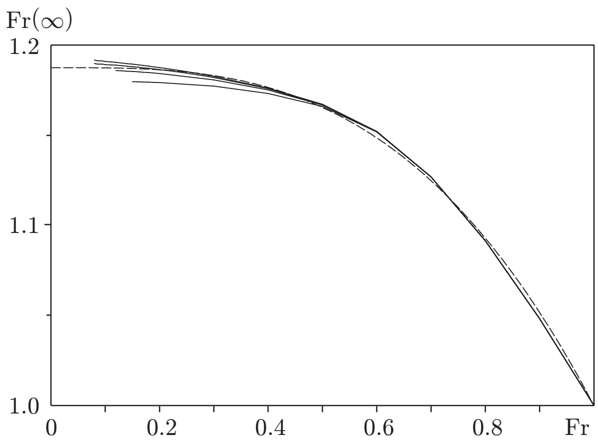

Fig. 5

Fig. 4. Curves of L versus H/h for $\beta = 0.2$, 0.3, 0.4, and 0.5: the dashed curve refers to calculation using the MFRP; the parametric dependence is denoted by points.

Fig. 5. Curves of $\mathrm{Fr}(\infty)$ versus Fr for various slopes of the step.

rate occurs for flows possessing waves at infinity on the right, and in the hydraulic treatment of the MFRP, the maximum flow rate occurs for fictitious waveless flows that obey Eq. (1).

For $H/h < 0$ (when the step depresses the bottom level) and $0 < \mathrm{Fr} < 1$, waveless flow regimes were not detected in [6]. The absence of waveless regimes for $H/h < 0$ is also confirmed using the MFRP because from (4) it follows that if $0 < \mathrm{Fr} < 1$, then $0 < H/h < 1$.

The goals of the calculations performed in the present study were:

— to quantitatively test the MFRP for slopes $0 < \beta \leqslant 1/2$;

— to elucidate whether the relation between Fr and H/h, which defines the waveless flow regime, is independent of the slope of the step;

— to refine, if possible, the approximate analytical relationship between Fr and H/h defined by formula (4);

— to find the free-boundary shapes.

Figure 3 shows curves of H/h versus Fr for various slopes of the step. Dependence (4) is also presented here. As can be seen, the curves practically merge with each other. Thus, the data in Fig. 3 suggest that the MFRP is approximately valid. Quantitative differences between the calculated values and the values obtained using the MFRP are clearly seen in Fig. 4, which shows curves of the free-surface height L at infinity on the right versus the step height H/h for various slopes of the step β. It is obvious that the MFRP gives overestimated values of L,

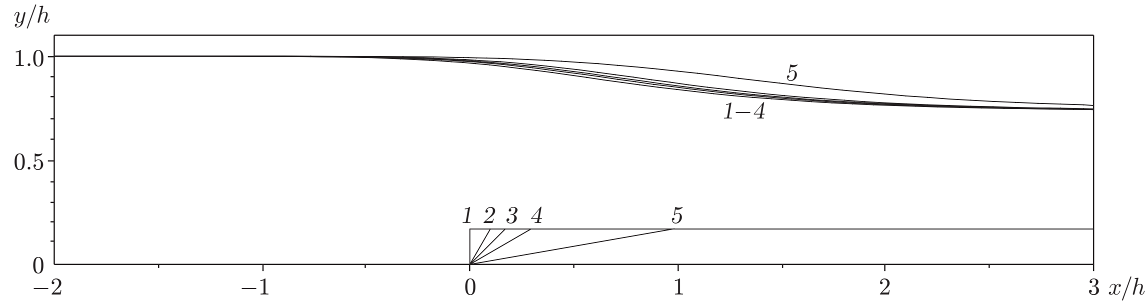

Fig. 6. Free-surface shapes for subcritical flows (Fr = 0.5) at $\beta = 1/2$ (1), 1/3 (2), 1/4 (3), 1/6 (4) and 1/18 (5).

and the maximum discrepancy is observed for $H/h \approx 0.15$. However, for all β, the dependences of L on H/h are very similar. This suggests that the MFRP can be refined. Indeed, according to the MFRP, the flow at infinity on the right is always critical, i.e, $\mathrm{Fr}(\infty) = 1$. However, the results of the calculations shows that this is not so. The number $\mathrm{Fr}(\infty)$ is always larger than unity, and when Fr tends to zero, the value of $\mathrm{Fr}(\infty)$ approaches a certain limiting value, which depends only slightly on the slope of the step and varies from $\mathrm{Fr}(\infty) = 1.1794$ for $\beta = 0.2$ to $\mathrm{Fr}(\infty) = 1.1913$ for $\beta = 0.5$. Figure 5 shows curves of $\mathrm{Fr}(\infty)$ versus Fr for various slopes of the step β.

As can be seen, the functions plotted in Fig. 5 depend weakly on the slope of the step β. Let

$$\mathrm{Fr}(\infty) = f(\mathrm{Fr}) = a_1(1 - \mathrm{Fr})\,\mathrm{Fr}^2 + a_2(1 - \mathrm{Fr}^2) + 1,$$

where the function $f(\mathrm{Fr})$ has properties $f'(0) = 0$ and $f(1) = 1$ for any a_1 and a_2. Using the least-squares method, we found the values $a_1 = 0.1982$ and $a_2 = 0.1871$. The function $f(\mathrm{Fr})$ for these values is shown in Fig. 5 by a dashed curve. It is evident that the calculated points are well approximated by the function. Thus, for any slopes of the step in the range $0.2 < \beta \leqslant 0.5$, we obtained an approximate analytical relation between Fr and $\mathrm{Fr}(\infty)$.

From formulas (1) and (2), we find that

$$H/h = 1 + [\mathrm{Fr}^2 - \mathrm{Fr}^{2/3}(\mathrm{Fr}^2(\infty) + 2)/\,\mathrm{Fr}^{2/3}(\infty)]/2,$$

$$L = [\mathrm{Fr}^2 + 2 - \mathrm{Fr}^{2/3}\,\mathrm{Fr}^{4/3}(\infty)]/2.$$

The parametric relation between H/h and L, denoted in Fig. 4 by points, gives much more accurate result than MFRP.

The free-surface shapes for various values of the slope of the step for Fr = 0.5 are given in Fig. 6.

Supercritical Flow. In the case where supercritical flow on the left becomes supercritical on the right, the step height in the physical plane (H) is specified beforehand. Calculations of supercritical flow over a vertical step increasing the bottom level were performed in [7]. Numerical solutions were found for arbitrary $H/h > 0$ that satisfy inequality (1). Probably, inequality (1) is a necessary and sufficient condition for the existence of a solution for which supercritical flow on the left becomes supercritical on the right. The calculations performed in the present study confirm this conclusion for an inclined step, too. In this case, $0 < H/h < 0.6188$. We note that for

$$H/h = (2 + \mathrm{Fr}^2 - 3\,\mathrm{Fr}^3)/2,$$

the flow becomes critical on the right $[\mathrm{Fr}(\infty) = 1]$, which prevents a further increase in the step height. The case $H/h < 0$ ($\beta < 0$) was not considered in [7]. Our calculation showed that a solution of the problem can be found for any $H/h < 0$.

The free boundaries for large values of H/h (flow of the type of a water fall) are shown in Fig. 7. In this case, the free-boundary segment along the step becomes extended and it is possible to obtain an approximate formula for the free-surface shape. Indeed, from Bernoulli's equation, we find that the velocity at an arbitrary point $B(x, y)$ on the free surface is calculated by the formula

$$V(y) = V_0 h/H(y),$$

where $H(y)$ is the layer width. It is easy to derive that

$$x = y \sin\beta\pi + H(y)/\sin\beta\pi.$$

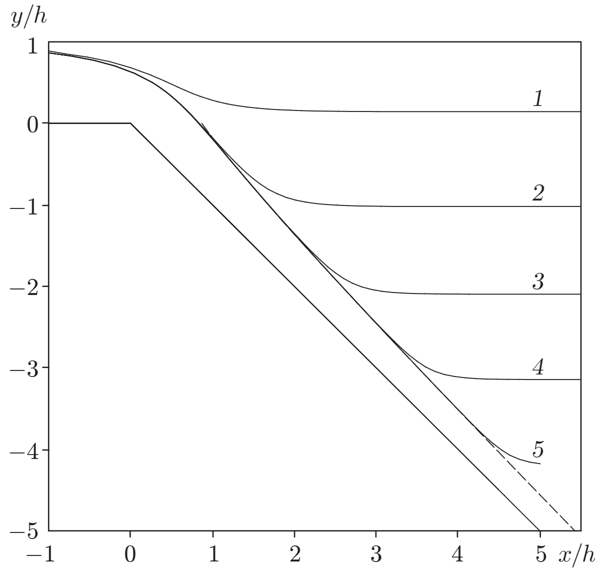

Fig. 7. Free-surface shapes in the case of supercritical flows (Fr = 1.1) at $\beta = -1/4$ for $H/h = -0.5$ (1), -1.5 (2), -2.5 (3), -3.5 (4), and -4.5 (5).

From this it follows that

$$\frac{x}{h} = \frac{y}{h}\sin\beta\pi + \frac{\mathrm{Fr}}{\sin\beta\pi\sqrt{\mathrm{Fr}^2 + 2(1 - y/h)}}.$$

The results of calculations using this approximate formula are shown in Fig. 7 by a dashed curve. As is evident from the figure, this curve approximates the free-surface shape well if the step height is great enough.

It is known that for supercritical flow over a step on a horizontal bottom in a certain range of Froude numbers, the problem can have two solutions [12, 13] and a larger number of solutions [14]. One of them branches from the solution for a uniform flow, and the remaining branch from a solitary wave. The existence of three and larger numbers of solutions is probably related to the oscillating behavior of the parameters of steep waves, which was first revealed in [15, 16] using an asymptotic analysis. In the exact nonlinear formulation, nearly limiting gravity waves were studied in [11]. From the results of [11] it follows, in particular, that it is always possible to specify the range of Froude numbers in which the solitary wave problem has any beforehand specified number of solutions. The question arises as to whether similar nonuniqueness takes place in the problem of flow over a step. To answer this question, the following numerical experiment was performed. It was assumed that $d = 0$ in Eq. (27), and a function $\lambda(\xi)$ was found that corresponded to a solitary wave. Then, a small parameter d was specified, the function $\lambda(\xi)$ obtained for the case $d = 0$ was used as a zero approximation, and Newton's method was applied to Eq. (27). The iterative process diverged for an arbitrary small d. Thus, it was concluded that the problem of flow over a step does not have a solution that branches from the solution for a solitary wave. This is probably due to the different bottom levels at infinity on the left and on the right of the step.

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